Critical phenomena from the two-particle irreducible $1/N$ expansion

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Abstract

The $1/N$ expansion of the two-particle irreducible (2PI) effective action is employed to compute universal properties at the second-order phase transition of an $O(N)$-symmetric N-vector model directly in three dimensions. At next-to-leading order the approach cures the spurious small-N divergence of the standard (1PI) $1/N$ expansion for a computation of the critical anomalous dimension $\eta(N)$, and leads to improved estimates already for moderate values of N.

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1 Introduction

New methods for the quantitative description of thermal equilibrium as well as nonequilibrium aspects of critical phenomena have a wide range of important applications. Topical examples are the active experimental searches and theoretical explorations of properties of the QCD critical point in the phase diagram of strongly interacting matter [1], or the description of the critical dynamics of Bose-Einstein condensates in laboratory experiments with ultracold quantum gases [2]. On extremely different energy scales their quantitative understanding requires a field theoretical description of the equilibrium and nonequilibrium properties in the vicinity of critical points associated to second-order phase transitions. The latter exhibit anomalously large fluctuations and are characterized by universal quantities such as critical exponents.

There are few nonperturbative methods in thermal equilibrium that can describe the large fluctuations in the vicinity of a second-order phase transition [3]. However, the number of methods becomes even more limited once nonequilibrium dynamics and the approach to thermal equilibrium is considered. It is important to note that mean-field-type or leading-order large-N approximations are insufficient. They are known to fail to describe thermalization and do not properly distinguish the different universality classes for critical phenomena. Efforts to go beyond leading order in a standard $1/N$ expansion of the one-particle irreducible (1PI) effective action fail because of spurious secular terms, which grow with time and invalidate the approximation for nonequilibrium physics. In addition, the 1PI $1/N$ expansion shows a rather poor convergence in thermal equilibrium for moderate values of N, and at low order important quantities such as the critical anomalous dimension $\eta(N)$ are known to exhibit a spurious divergence as $N \to 0$ [4].

It has recently been pointed out that a controlled description of nonequilibrium dynamics and thermalization of quantum fields can be based on a $1/N$ expansion of the two-particle irreducible (2PI) effective action beyond leading order [5, 6, 7, 8, 9, 10]. For nonequilibrium dynamics the rapid convergence of this expansion has also been observed in classical statistical field theories, where comparisons with exact results are possible [11]. This nonperturbative approach provides a promising candidate for a uniquely suitable description of *both* the nonequilibrium as well as equilibrium physics in the vicinity of critical points, since it provides a controlled expansion

even in the presence of large fluctuations. An important step towards such a conclusion is, therefore, to show that the 2PI $1/N$ expansion indeed reliably describes the thermal equilibrium properties at the critical point of a second-order phase transition. In this work we calculate universal properties, employing the 2PI $1/N$ expansion to next-to-leading order (NLO). We consider the $O(N)$-symmetric scalar N-vector model in three-dimensions. For $N = 4$ this model is expected to belong to the universality class of the high temperature QCD phase transition in the limit of two massless quark flavors [12], and for $N = 1$ to the QCD critical point at high baryon density and temperature [13]. In the context of Bose-Einstein condensation in quantum gases $N = 2$ characterizes the relevant universality class [2]. Here it is important to note that the equilibrium universality classes agree in the relativistic and the nonrelativistic case, and we will only consider the former.

To show the capabilities of the method we calculate the properties of the theory directly at the critical temperature of a second-order phase transition. This is notoriously difficult within perturbative approaches since the correlation lengths are diverging. All the universal properties at this point are encoded in a single critical exponent, which can be associated to the anomalous dimension η. In this case the knowledge of η also fixes all universal information about the effective potential, which encodes the information about the critical equation of state. We show that the 2PI $1/N$ expansion cures the spurious small-N divergence of the standard (1PI) $1/N$ expansion, and leads to improved estimates already for moderate values of N. We finally compare the method with related approaches that have been employed in the literature.

2 Model

We consider a Euclidean field theory for a real, N–component scalar field φ_a ($a = 1, \ldots, N$) with classical action

$$S[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2} \partial_\mu \varphi_a(x) \partial_\mu \varphi_a(x) + \frac{m^2}{2} \varphi_a(x)\varphi_a(x) + \frac{\lambda}{4! N} \left(\varphi_a(x)\varphi_a(x)\right)^2 \right) , \tag{2.1}$$

where summation over repeated indices is implied. All information about the equilibrium field theory can be obtained from the partition function, or more efficiently from an n-particle irreducible effective action [14], which are related by Legendre transforms in the presence of sources. Here we consider

the two-particle irreducible (2PI) representation of the effective action [15]. The most general Euclidean 2PI effective action can be written as

$$\Gamma[\phi, G] = S[\phi] + \frac{1}{2}\mathrm{Tr}\ln G^{-1} + \frac{1}{2}\mathrm{Tr}\, G_0^{-1}(\phi)\, G + \Gamma_2[\phi, G] + \mathrm{const}\,. \qquad (2.2)$$

Diagrammatically the contribution $\Gamma_2[\phi, G]$ is given by all two-particle irreducible[1] graphs with propagator lines set equal to G [15]. The classical inverse propagator $G^{-1}_{0,ab}(x, y; \phi) = \delta^2 S[\phi]/\delta\phi_a(x)\delta\phi_b(y)$ reads

$$\begin{aligned} G^{-1}_{0,ab}(x, y; \phi) \;\; &= \;\; \left(-\partial_\mu\partial_\mu + m^2 + \frac{\lambda}{6N}\,\phi_c(x)\phi_c(x)\right)\delta_{ab}\delta^d(x-y) \\ &\quad + \frac{\lambda}{3N}\,\phi_a(x)\phi_b(x)\delta^d(x-y)\,. \end{aligned} \qquad (2.3)$$

In the presence of an external source $J_a(x)$ coupling linearly to the fluctuating field, the equations of motion for ϕ and G are [15]

$$\frac{\delta\Gamma[\phi, G]}{\delta\phi_a(x)} = J_a(x) \quad , \quad \frac{\delta\Gamma[\phi, G]}{\delta G_{ab}(x, y)} \;=\; 0\,. \qquad (2.4)$$

3 Two-particle irreducible $1/N$ expansion

In this work we consider a systematic expansion of the 2PI effective action $\Gamma[\phi, G]$ in the number of field components or powers of $1/N$ beyond leading order [5, 6]. We write

$$\Gamma_2[\phi; G] = \Gamma_2^{\mathrm{LO}}[G] + \Gamma_2^{\mathrm{NLO}}[\phi; G] + \Gamma_2^{\mathrm{NNLO}}[\phi; G] + \ldots \qquad (3.1)$$

where Γ_2^{LO} denotes the leading order (LO) contribution which scales proportional to N, while Γ_2^{NLO} is the next-to-leading order (NLO) contribution $\sim N^0$, and $\Gamma_2^{\mathrm{NNLO}}[G] \sim 1/N$ etc. For the $O(N)$–model these contributions have been derived in Ref. [5, 6].[2] The LO and NLO contributions, which we will consider here, read (cf. Fig. 1)

[1] A diagram is said to be two-particle irreducible if it does not become disconnected by opening two lines.

[2] Note that in Ref. [5, 6] a Minkowskian space-time is considered, whereas here a Euclidean metric is used.

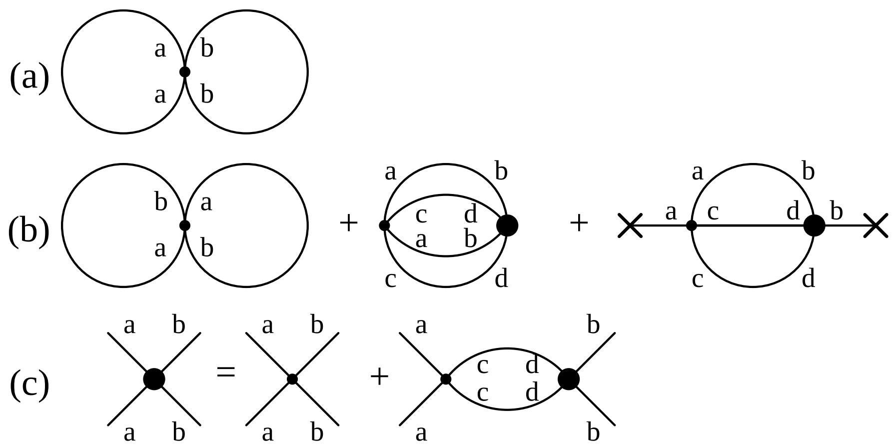

Figure 1: Diagrammatic representation of the LO and NLO contributions to $\Gamma_2[\phi, G]$. The solid lines and small dots represent the full propagator and the bare vertex respectively and the indices run from $a = 1, \ldots, N$. Diagram (a) is $\Gamma_2^{\rm LO}$ given by (3.2). Diagrams (b) and (c) are $\Gamma_2^{\rm NLO}$, where we have expressed this contribution as a three-loop diagram with an effective four-vertex containing a "chain" of bubbles. This form for $\Gamma_2^{\rm NLO}$ can been seen from (3.3) with the aid of (3.5) and (3.6).

$$
\begin{aligned}
\Gamma_2^{\rm LO}[G] &= \frac{\lambda}{4!N}\int_x G_{aa}(x,x)G_{bb}(x,x)\,, && (3.2)\\
\Gamma_2^{\rm NLO}[\phi, G] &= \frac{1}{2}{\rm Tr}\ln[\,\mathbf{B}(G)\,] \\
&- \frac{\lambda}{6N}\int_{xy}\mathbf{I}(x,y;G)\phi_a(x)G_{ab}(x,y)\phi_b(y). && (3.3)
\end{aligned}
$$

In the above equation we have defined

$$
\mathbf{B}(x,y;G) = \delta^d(x-y) + \frac{\lambda}{6N}G_{ab}(x,y)G_{ab}(x,y), \tag{3.4}
$$

and the logarithm in Eq. (3.3) sums the infinite series

$$
\begin{aligned}
{\rm Tr}\ln[\,\mathbf{B}(G)\,] = \int_x \left(\frac{\lambda}{6N}G_{ab}(x,x)G_{ab}(x,x)\right) \qquad (3.5)\\
-\frac{1}{2}\int_{xy}\left(\frac{\lambda}{6N}G_{ab}(x,y)G_{ab}(x,y)\right)\left(\frac{\lambda}{6N}\,G_{a'b'}(y,x)G_{a'b'}(y,x)\right) + \ldots
\end{aligned}
$$

The function $\mathbf{I}(x,y;G)$ is defined by

$$\mathbf{I}(x,y;G) = \frac{\lambda}{6N} G_{ab}(x,y) G_{ab}(x,y) - \frac{\lambda}{6N} \int_z \mathbf{I}(x,z;G) G_{ab}(z,y) G_{ab}(z,y)\,, \tag{3.6}$$

and resums an infinite number of "chain" graphs, which can be seen by iteratively expanding (3.6). The function $\mathbf{I}(x,y;G)$ and the inverse of $\mathbf{B}(x,y;G)$ are related by

$$\mathbf{B}^{-1}(x,y;G) = \delta^d(x-y) - \mathbf{I}(x,y;G)\,, \tag{3.7}$$

which follows from convoluting Eq. (3.4) with $\mathbf{B}^{-1}$ and using Eq. (3.6). Note that $\mathbf{B}$ and $\mathbf{I}$ do not depend on ϕ, and $\Gamma_2[\phi, G]$ is only quadratic in ϕ at NLO.

4 Universal behavior at the critical point

In the following we concentrate on $d = 3$, relevant for high temperature quantum field theories or classical statistical models with three spacial dimensions. For this case the model is known to exhibit a second-order phase transition at a critical mass parameter $m^2 = m_c^2(N, \lambda)$ for all N. In Fourier space we will use the notation $\int \mathrm{d}^3q/(2\pi)^3 \equiv \int_{\mathbf{q}}$.[3] To show the capabilities of the method we calculate the properties of the theory directly for the critical value m_c^2. In this case the correlation lengths are diverging, which spoils standard perturbative approaches. All the universal properties at this point are encoded in a single critical exponent.[4] In the absence of external sources this exponent can be associated to the anomalous dimension η. It characterizes the critical behavior of the propagator or two-point function:

$$G(\mathbf{p}) = \frac{1}{\mathbf{p}^2} \left(\frac{\mathbf{p}^2}{\Lambda^2} \right)^{\eta/2}\,, \tag{4.1}$$

which is valid in the limit $\mathbf{p}^2/\Lambda^2 \to 0^+$. Here Λ corresponds to an (arbitrary) high momentum scale which regularizes the theory. All universal properties are independent of Λ as is also shown below.

[3] If not stated otherwise most formulae are valid for general d and the restriction to three dimensions will only be relevant for some specific momentum integrals below.

[4] Note that deviations from the critical point are described by a further independent critical exponent.

We note that the knowledge of η also fixes all universal information about the effective potential $U(\phi)$, which encodes the complete information about the critical equation of state in the presence of an external (static) source $\sim J_a$ (cf. (2.4) and below). At the critical point one has

$$\frac{\partial U(\phi)}{\partial \phi} \sim \phi^{\delta}\,, \tag{4.2}$$

valid for $\phi \to 0$ with a nonuniversal proportionality constant that depends on the specific details of the model. Here the universal critical exponent δ is not independent but related to η by the scaling relation [3]

$$\delta = \frac{5-\eta}{1+\eta} \tag{4.3}$$

for $d = 3$. The presence of scaling relations such as (4.3) is a consequence of the limited number of relevant parameters, which can regulate the diverging correlation lengths at the critical point. For the $O(N)$–model there is only one independent relevant parameter at the critical point for $m^2 = m_c^2$, which can either be associated to a nonvanishing momentum or source. As a consequence, one may either extract the universal behavior from (4.1) or from (4.2). We emphasize that the scaling relation (4.3) is a robust property of the theory in the absence of additional relevant parameters. In contrast, the error involved in an approximate estimate for a difficult quantity such as the anomalous dimension is typically large. In the following we derive the relevant equations for both η and δ, employing the 2PI $1/N$ expansion to NLO. We will then choose the simpler (former) case for an explicit solution of the equations in order to obtain a quantitative estimate of the universal behavior.

Firstly, for a computation of the effective potential it is sufficient to consider a constant field expectation value. By virtue of $O(N)$ rotations the most general field configuration in this case can be chosen as

$$\phi_a(x) \;=\; \sqrt{\frac{6N}{\lambda}}\,\phi\,\delta_{a1}\,, \tag{4.4}$$

$$G_{ab}(x,y) \;=\; \mathrm{diag}\left\{G_{\parallel}(x-y), G_{\perp}(x-y), \ldots, G_{\perp}(x-y)\right\}\,, \tag{4.5}$$

where we have rescaled the field for later convenience. For $N > 1$ the stationarity condition (2.4) for the composite field G with (3.2) and (3.3)

translates into two coupled equations for the longitudinal and transverse components. In Fourier space one finds

$$\begin{aligned} G_{\parallel}^{-1}(\mathbf{p}) &= \mathbf{p}^2 + m^2 + 3\phi^2 + \int_{\mathbf{q}} \left\{ \frac{\lambda}{6N} \left[3G_{\parallel}(\mathbf{q}) + (N-1)G_{\perp}(\mathbf{q}) \right] \right. \\ &- \left. 2\phi^2\, \mathbf{I}(\mathbf{q}) - \frac{\lambda}{3N} \left[\mathbf{I}(\mathbf{q}) + 2\phi^2 G_{\parallel}(\mathbf{q}) \left(1 - \mathbf{I}(\mathbf{q})\right)^2 \right] G_{\parallel}(\mathbf{p}-\mathbf{q}) \right\}, \\ G_{\perp}^{-1}(\mathbf{p}) &= \mathbf{p}^2 + m^2 + \phi^2 + \int_{\mathbf{q}} \left\{ \frac{\lambda}{6N} \left[G_{\parallel}(\mathbf{q}) + (N+1)G_{\perp}(\mathbf{q}) \right] \right. \\ &- \left. \frac{\lambda}{3N} \left[\mathbf{I}(\mathbf{q}) + 2\phi^2 G_{\parallel}(\mathbf{q}) \left(1 - \mathbf{I}(\mathbf{q})\right)^2 \right] G_{\perp}(\mathbf{p}-\mathbf{q}) \right\}. \end{aligned} \tag{4.6}$$

Here the resummation function $\mathbf{I}$ is given by (3.6) for the configuration (4.5), in Fourier space. We will discuss the resummation function in more detail below.[5] The effective potential $U(\phi)$ is determined by the effective action for a constant field:

$$U(\phi) = \frac{1}{V_3}\, \Gamma[\phi, G(\phi)]|_{\phi=\mathrm{const}}\ , \tag{4.8}$$

where V_3 is the three-dimensional volume. Here $G(\phi) = \{G_{\parallel}(\phi), G_{\perp}(\phi)\}$ denotes the solutions of (4.6). We then obtain from (2.2) with (3.2)–(3.3) the relevant equation for determining the equation of state:

$$\begin{aligned} \frac{\partial U(\phi)}{\partial \phi} &= \frac{6N\phi}{\lambda} \Big[m^2 + \phi^2 + \frac{\lambda}{6N} \int_{\mathbf{q}} \Big\{ 3G_{\parallel}(\mathbf{q};\phi) + (N-1)G_{\perp}(\mathbf{q};\phi) \\ & \quad -2\,\mathbf{I}(\mathbf{q}) G_{\parallel}(-\mathbf{q};\phi) \Big\} \Big]. \end{aligned} \tag{4.9}$$

We emphasize that the equations (4.6) and (4.9) are valid also away from the critical point of a second-order phase transition. In particular, the effective potential can be used to compute the different longitudinal and transverse susceptibilities in the phase with spontaneous symmetry breaking for $N > 1$:

$$\chi_{\parallel}^{-1} \sim \frac{\partial^2 U(\phi)}{\partial\phi\partial\phi} \qquad , \qquad \chi_{\perp}^{-1} \sim \frac{1}{\phi}\frac{\partial U}{\partial \phi}\, . \tag{4.10}$$

[5] The equation of motion for the field expectation value is not required since one considers $U(\phi)$ for all ϕ. For completeness we note that the stationarity condition for the field (2.4) leads for $\phi \neq 0$ to

$$\phi^2 = -m^2 - \frac{\lambda}{6N} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \left[3G_{\parallel}(q) + (N-1)G_{\perp}(q) - 2\,\mathbf{I}(q) G_{\parallel}(-q) \right]. \tag{4.7}$$

Note that Goldstone's theorem is fulfilled for the 2PI $1/N$ expansion, which has been pointed out previously in Ref. [6]. To compute the universal properties at the critical point, i.e. to obtain the exponent δ from (4.2), one has to specify in (4.9) the critical value of the mass parameter $m^2 = m_c^2$. The latter can be obtained for $\phi = 0$ from either equation of (4.6) by the condition $G_{\parallel}^{-1}(\mathbf{p} = 0) \equiv G_{\perp}^{-1}(\mathbf{p} = 0) = 0$.

An alternative calculation of the universal properties at the critical point determines the anomalous dimension η. For our current purposes it is sufficient to perform this calculation explicitly, which is a considerably simpler task. By virtue of $O(N)$ rotations, at the critical point the most general field configuration in the absence of sources is given by

$$G_{ab}(\mathbf{p}) = G(\mathbf{p})\,\delta_{ab}\,. \tag{4.11}$$

Equations (4.6) then reduce to the single expression

$$G^{-1}(\mathbf{p}) = \mathbf{p}^2 + m^2 + \Sigma(\mathbf{p}) \tag{4.12}$$

where the self-energy contains a momentum-independent $\mathcal{O}(N^0)$ part and both momentum-independent and momentum-dependent $\mathcal{O}(1/N)$ parts,

$$\Sigma(\mathbf{p}) = \lambda\,\frac{N+2}{6N}\int_{\mathbf{q}} G(\mathbf{q}) - \frac{\lambda}{3N}\int_{\mathbf{q}} G(\mathbf{p}-\mathbf{q})\,\mathbf{I}(\mathbf{q}) \tag{4.13}$$

and the chain sum is

$$\mathbf{I}(\mathbf{q}) = 1 - \left(1 + \frac{\lambda}{6}\int_{\mathbf{k}} G(\mathbf{q}-\mathbf{k})G(\mathbf{k})\right)^{-1}\,. \tag{4.14}$$

The equivalence of the explicit form for the resummation function (4.14) with the implicit form of Eq. (3.6) for the configuration (4.11) can be observed by expansion in a geometric series. Eqs. (4.12)–(4.14) provide a closed set of self–consistent equations to determine the full propagator $G(\mathbf{p})$. Fig. 2 shows them in diagrammatic form.

According to (4.1) at the critical point the inverse propagator at zero momentum vanishes, i.e. $G^{-1}(\mathbf{p} = 0) = 0$. Using this and subtracting from (4.12) the same expression for zero momentum we can write

$$G^{-1}(\mathbf{p}) = \mathbf{p}^2 - \frac{\lambda}{3N}\int_{\mathbf{q}} \left(G(\mathbf{p}-\mathbf{q}) - G(\mathbf{q})\right)\mathbf{I}(\mathbf{q})\,. \tag{4.15}$$

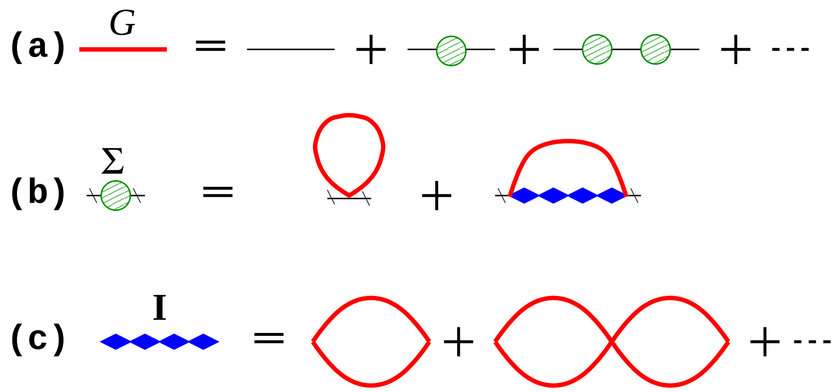

Figure 2: Diagrammatic representation of self-consistent equations for the full propagator $G(p)$ obtained from a $1/N$ expansion of the 2PI effective action to NLO for $\phi = 0$. Diagram (a) is (4.12), which expresses the full propagator in terms of the classical propagator G_0 (thin line) and the self-energy $\Sigma(p)$ (shaded blob). Diagram (b) is (4.13), which shows how the self-energy, the amputated one-particle-irreducible two-point function, can be expressed in terms of a one loop contribution containing the LO contribution, and a NLO contribution that involves the chain sum. Diagram (c), corresponding to (4.14), shows how the chain sum can be expressed in terms of the full propagator.

This equation is valid at the critical point and can be conveniently employed to extract universal properties. With the help of (4.14) this gives

$$G^{-1}(\mathbf{p}) = \mathbf{p}^2 + \frac{\lambda}{3N} \int_{\mathbf{q}} (G(\mathbf{p}-\mathbf{q}) - G(\mathbf{q})) \left(1 + \frac{\lambda}{6} \int_{\mathbf{k}} G(\mathbf{q}-\mathbf{k}) G(\mathbf{k})\right)^{-1}, \tag{4.16}$$

where we used that the integral involving the constant part of $\mathbf{I}(\mathbf{q})$ vanishes for $\mathbf{p}^2/\Lambda^2 \to 0$. The latter limit characterizes the momenta for which the universal exponent η of the critical form for the propagator (4.1) is defined. It remains to be shown that (4.1) indeed solves the above equation in this limit. To show this, we insert (4.1) into (4.16) and obtain an equation for the anomalous dimension η. The latter equation has a unique solution for given N as is demonstrated in the following.

Using (4.1) the one-loop subintegral in Eq. (4.16) can be performed for

$-1 < \eta < 1/2$ to give[6]

$$
\begin{aligned}
\frac{\lambda}{6}\int_{\mathbf{k}} G(\mathbf{q}-\mathbf{k})G(\mathbf{k}) &= \frac{\lambda}{6\Lambda}\int_{\mathbf{k}/\Lambda}\left(\frac{(\mathbf{q}-\mathbf{k})^2}{\Lambda^2}\right)^{\eta/2-1}\left(\frac{\mathbf{k}^2}{\Lambda^2}\right)^{\eta/2-1} \\
&= \frac{\lambda}{6\Lambda}\left(\frac{\mathbf{q}^2}{\Lambda^2}\right)^{\eta-1/2}\mathcal{A}(\eta)\,, \qquad (4.17)
\end{aligned}
$$

with

$$
\mathcal{A}(\eta) = \frac{1}{8\pi^{3/2}}\,\frac{\Gamma(\frac{1}{2}-\eta)\left(\Gamma(\frac{1+\eta}{2})\right)^2}{\left(\Gamma(1-\frac{\eta}{2})\right)^2\Gamma(1+\eta)}\,. \qquad (4.18)
$$

With this notation the equation (4.16) takes the form

$$
\begin{aligned}
\left(\frac{\mathbf{p}^2}{\Lambda^2}\right)^{1-\eta/2} &= \frac{\mathbf{p}^2}{\Lambda^2} + \frac{\lambda}{3N\Lambda}\int_{\mathbf{q}/\Lambda}\left(\left(\frac{(\mathbf{p}-\mathbf{q})^2}{\Lambda^2}\right)^{\eta/2-1} - \left(\frac{\mathbf{q}^2}{\Lambda^2}\right)^{\eta/2-1}\right) \\
&\times \left(1 + \frac{\lambda}{6\Lambda}\left(\frac{\mathbf{q}^2}{\Lambda^2}\right)^{\eta-1/2}\mathcal{A}(\eta)\right)^{-1}\,. \qquad (4.19)
\end{aligned}
$$

This provides a self-consistency equation for η which may be solved numerically for given N and λ/Λ. We have done this for a check which is discussed below.

However, one can proceed further analytically to obtain η. For the low momentum range of critical phenomena, $\mathbf{p}^2/\Lambda^2 \to 0^+$, the remaining momentum integral is dominated by small $\mathbf{q}^2 \sim \mathbf{p}^2$ (cf. above that $\eta < 1/2$ for the allowed range). In this limit we can, therefore, write

$$
\left(\frac{\mathbf{p}^2}{\Lambda^2}\right)^{1-\eta/2} = \frac{\mathbf{p}^2}{\Lambda^2} + \frac{2}{\mathcal{A}(\eta)N}\int_{\mathbf{q}/\Lambda}\left(\left(\frac{(\mathbf{p}-\mathbf{q})^2}{\Lambda^2}\right)^{\eta/2-1} - \left(\frac{\mathbf{q}^2}{\Lambda^2}\right)^{\eta/2-1}\right)\left(\frac{\mathbf{q}^2}{\Lambda^2}\right)^{1/2-\eta}. \qquad (4.20)
$$

[6] It can be conveniently obtained using the Feynman parametrization for non-integer exponents α, β:

$$
\frac{1}{A^\alpha B^\beta} = \int_0^1 dx_1 dx_2\,\delta(x_1+x_2-1)\frac{x_1^{\alpha-1}x_2^{\beta-1}}{(x_1 A + x_2 B)^{\alpha+\beta}}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,.
$$

The momentum integration can then be performed for $\eta < 1/2$, and the subsequent integration over the Feynman parameter for $\eta > -1$ with the result (4.17). Note that we have evaluated the integral without regulator. One can show that subleading corrections obtained from keeping the momentum integration finite are irrelevant for the universal low momentum behavior.

One observes that the dependence on the coupling λ dropped out completely from the equation. This is a manifestation of universality, which implies that the anomalous dimension for the three-dimensional $O(N)$–model is only a function of the number of field components N. After performing an elementary angle integration one finds with $\eta \neq 0$ and using the notation $p \equiv |\mathbf{p}|/\Lambda$ and $q \equiv |\mathbf{q}|/\Lambda$:

$$p^{2-\eta} = p^2 + \frac{1}{2\pi^2 \mathcal{A}(\eta) N} \int_0^1 dq \left(\frac{q^{2-2\eta}}{\eta p} \left(\left([p+q]^2\right)^{\eta/2} - \left([p-q]^2\right)^{\eta/2} \right) - 2q^{1-\eta} \right). \tag{4.21}$$

The remaining momentum integral can be performed with the help of hypergeometric functions. This is described in the appendix. The result can be written as a sum of an anomalous term $\sim p^{2-\eta}$ and a regular function $F_\eta(p^2)$ of momentum squared:

$$p^{2-\eta} = p^2 + \mathcal{B}(\eta)\, p^{2-\eta} + F_\eta(p^2)\,, \tag{4.22}$$

with

$$\mathcal{B}(\eta) \;=\; \frac{4\eta(1-2\eta)\cos(\eta\pi)}{(3-\eta)(2-\eta)\sin^2(\eta\pi/2) N} + \mathcal{O}\left(\frac{1}{N^2}\right) . \tag{4.23}$$

The regular function can be expanded in powers of the critical momentum $p \to 0$ as

$$F_\eta(p^2) = -\frac{(1-\eta)(2-\eta)}{6\pi^2 \eta \mathcal{A}(\eta) N}\, p^2 + \mathcal{O}\left(p^4, \frac{1}{N^2}\right) . \tag{4.24}$$

Since the behavior of $F_\eta(p^2)$ for small momenta is $\sim p^2$, this function along with the p^2-term in (4.22) are subleading for a positive anomalous dimension. In this case, the $p \to 0$ behavior is dominated by the anomalous term $\sim p^{2-\eta}$. Comparing the left and the right hand side of Eq. (4.22), one observes that η has to fulfill

$$\mathcal{B}(\eta) \overset{!}{=} 1\,. \tag{4.25}$$

We observe from this constraint with (4.23) that the anomalous dimension indeed has to be positive for the allowed range $\eta < 1/2$, and (4.25) can be used to extract the critical exponent η. In addition, we have numerically determined η directly from Eq. (4.19) for selective values of N, which agree.[7]

[7] We note that an approximate estimate for η may also be obtained from enforcing that

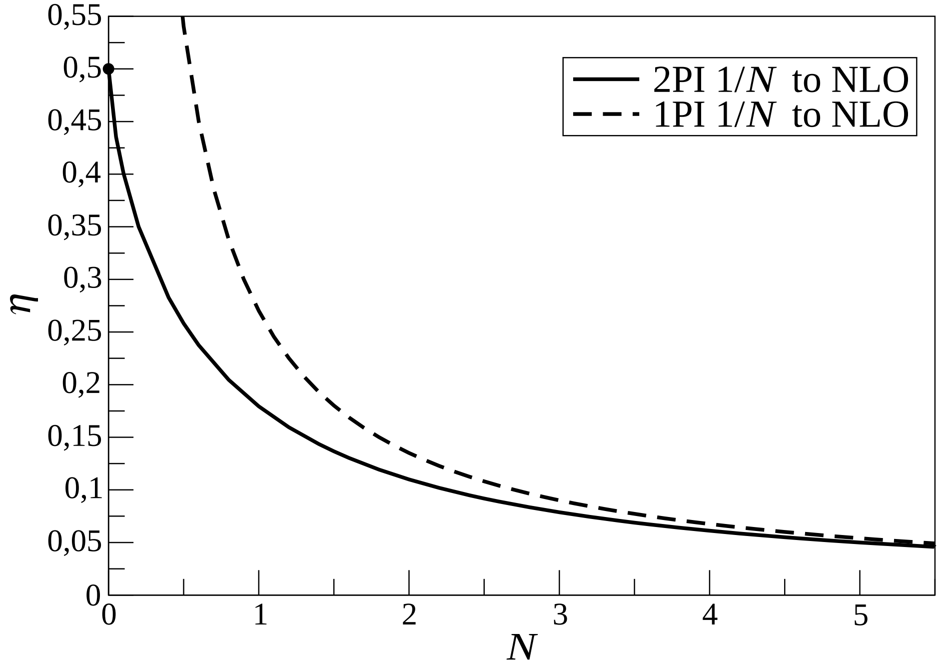

Figure 3: The anomalous dimension η for the three-dimensional $O(N)$–model as a function of N. The figure compares the results obtained from the 2PI $1/N$ expansion to next-to-leading order (NLO) with the conventional 1PI $1/N$ expansion to NLO. One observes that the expansion of the 2PI effective action cures the spurious $N \to 0$ divergence of the 1PI $1/N$ expansion. This is reflected in the improved estimates of η for small N if compared to results from alternative methods [3]. In contrast, for large enough N both expansions converge to the same results and $\lim\limits_{N\to\infty} \eta(N) = 0$.

In Fig. 3 we have displayed the results for η obtained from (4.25) as a function of N (solid line). For comparison, we also show the corresponding results from the conventional (1PI) $1/N$ expansion to NLO (dashed line) [4]. In contrast to the latter, one observes that the expansion of the 2PI effective action leads to a well-defined limit $\eta \to 0.5^-$ as $N \to 0^+$.[8] It therefore cures the spurious small-N divergence of the standard (1PI) $1/N$ expansion, which is reflected in the improved estimates of η for moderate values of N. Despite this qualitative and quantitative improvement, the NLO approximation within the 2PI $1/N$ expansion still cannot compete with the more elaborate estimates from alternative methods for small N. For instance, the 2PI result for $N = 4$ is $\eta = 0.061$, which is still about 35–40% too high compared to alternative estimates [3].

In the following we compare the 2PI $1/N$ expansion with previous approximations based on an ansatz for the exact Schwinger-Dyson equation for the propagator. We note that the result from the 2PI $1/N$ expansion at NLO (4.25) agrees with the expression obtained by Bray [16] from his "Self-Consistent Screening Approximation" (SCSA). The latter approximation scheme corresponds also to the so-called "Bare Vertex Approximation" (BVA), which has been employed in the context of time evolution problems [7]. For BVA/SCSA one introduces an auxiliary field for a composite operator into the $O(N)$-model. In addition to the original propagator G, the effective theory then contains a propagator for the composite field and a two-point function mixing the original and composite field. The exact Schwinger-Dyson equations for the two-point functions are then approximated by keeping the interactions of the corresponding effective theory bare. As has been pointed out in Ref. [6], the BVA is not consistent with the $1/N$ expansion of the 2PI effective action in the presence of a source term or for $\phi \neq 0$, since BVA sums NLO and only part of the NNLO contributions. For $\phi \equiv 0$, the 2PI $1/N$ expansion to NLO and the BVA/SCSA

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the subleading terms $\sim p^2$ in (4.22) cancel, which leads with (4.24) to the condition

$$(1-\eta)(2-\eta) = 6\pi^2 \eta \mathcal{A}(\eta) N \,.$$

The solution of this equation and of (4.25) agrees to good accuracy for $N \gtrsim 1$, and have the same limit $\eta \to 0.5^-$ as $N \to 0^+$. However, only the leading contributions for $p \to 0^+$ were calculated properly since they determine the critical behavior, and this is described by the solutions of (4.25).

[8]The limit $N \to 0$ describes the universality class for the critical swelling of long polymer chains [17].

are identical, which is the reason for the agreement observed above. We emphasize that the 2PI $1/N$ expansion can be systematically improved and going beyond this order requires to include the NNLO corrections described in Ref. [5, 6].

5 Conclusions

The 2PI $1/N$ expansion represents one of the few nonperturbative methods that can calculate critical behavior directly in three dimensions. In particular, there are no improvement procedures involved such as Borel transformation and conformal mapping underlying results from expansions in $4 - \epsilon$ dimensions (cf. the first Ref. of [3]). In view of the improved behavior of the 2PI $1/N$ expansion as compared to the standard (1PI) $1/N$ expansion, the former seems to provide a promising candidate for quantitative estimates. Here it is important that the $1/N$ expansion of the 2PI effective action can be systematically improved. A very interesting further step would take into account the NNLO corrections as described in Ref. [5, 6]. But already the NLO approximation provides a valuable quantitative tool for studying critical phenomena. This concerns in particular real-time properties of quantum field theories. It should be stressed that the 2PI $1/N$ expansion presents, so far, a uniquely suitable method that can deal with both nonequilibrium [5, 6, 7, 8, 9, 10] as well as equilibrium problems even in the presence of large fluctuations. Our results are very encouraging for an application of these methods to dynamical properties of critical phenomena.

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6 Appendix

In this appendix we discuss the analytical evaluation of the momentum integral in Eq. (4.21). Writing the equation as

$$p^{2-\eta} = p^2 + \frac{1}{2\pi^2 \mathcal{A}(\eta) N} \left(R_1 + R_2 + R_3\right) \tag{6.1}$$

we decompose the integral into three terms. The first integral is

$$R_1 = \frac{p^{\eta-1}}{\eta} \int_0^1 dq\, q^{2-2\eta} \left(1 + \frac{q}{p}\right)^{\eta} , \tag{6.2}$$

which in terms of the hypergeometric function

$${}_2F_1(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dx\, \frac{x^{b-1}(1-x)^{c-b-1}}{(1-xz)^a} \tag{6.3}$$

becomes

$$R_1 \;=\; \frac{p^{\eta-1}}{\eta} \frac{1}{3-2\eta}\; {}_2F_1\left(3-2\eta, -\eta, 4-2\eta, -\frac{1}{p}\right) , \tag{6.4}$$

where we have simplified products of Γ-functions and relied upon the relation

$${}_2F_1(a, b, c, z) = {}_2F_1(b, a, c, z). \tag{6.5}$$

The hypergeometric function can be transformed further using [18]

$$\begin{aligned} {}_2F_1\left(a, b, c, z\right) = {} & \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)} (-z)^{-a}\; {}_2F_1\left(a, 1+a-c, 1+a-b, \frac{1}{z}\right) \\ & + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)} (-z)^{-b}\; {}_2F_1\left(b, 1+b-c, 1+b-a, \frac{1}{z}\right) . \end{aligned} \tag{6.6}$$

This yields hypergeometric functions suitable for a small p expansion, i.e.

$$\begin{aligned} {}_2F_1\left(3-2\eta, -\eta, 4-2\eta, -\frac{1}{p}\right) \;=\; & p^{3-2\eta} \left(\frac{\Gamma(4-2\eta)\,\Gamma(-3+\eta)}{\Gamma(-\eta)}\right) \\ & + p^{-\eta} \left(\frac{3-2\eta}{3-\eta} + \frac{\eta\,(3-2\eta)}{(2-\eta)}\, p - \frac{\eta\,(3-2\eta)}{2}\, p^2 \right. \\ & \left. - \frac{(3-2\eta)\;(1-\eta)\;(2-\eta)}{6}\, p^3 + \mathcal{O}\left(p^4\right)\right). \end{aligned} \tag{6.7}$$

The second integral is

$$R_2 = -\frac{p^{\eta-1}}{\eta}\int_0^1 dq\, q^{2-2\eta}\left(\left[1-\frac{q}{p}\right]^2\right)^{\eta/2}, \tag{6.8}$$

which may be evaluated with the aid of the following relation where $z > 1$:

$$\int_0^1 dx\, x^a(1-xz)^{2b} = z^{-1-a}\Bigg(\frac{\Gamma(-1-a-2b)\Gamma(1+2b)}{\Gamma(-a)} \tag{6.9}$$
$$+\frac{\Gamma(1+a)\Gamma(1+2b)}{\Gamma(2+a+2b)}\Bigg)+\frac{z^{2b}}{(1+a+2b)}\ {}_2F_1(-1-a-2b,-2b,-a-2b,1/z)\,.$$

So we obtain

$$\begin{aligned} R_2 &= -\frac{p^{2-\eta}}{\eta}\left(\frac{\Gamma(-3+\eta)\Gamma(1+\eta)}{\Gamma(-2+2\eta)}+\frac{\Gamma(3-2\eta)\Gamma(1+\eta)}{\Gamma(4-\eta)}\right) \\ &- \frac{1}{\eta p}\frac{1}{(3-\eta)}\ {}_2F_1(-3+\eta,-\eta,-2+\eta,p)\,. \end{aligned} \tag{6.10}$$

Expanding the hypergeometric function in small p one finds

$$\begin{aligned} {}_2F_1\left(-3+\eta,-\eta,-2+\eta,p\right) &= 1-\frac{\eta\,(3-\eta)}{2-\eta}\,p \\ - \quad \frac{\eta\,(3-\eta)}{2}\,p^2 &+ \frac{(3-\eta)\ (1-\eta)\ (2-\eta)}{6}\,p^3+\mathcal{O}\left(p^4\right)\,. \end{aligned} \tag{6.11}$$

The third integral is straightforward and gives

$$R_3 = -\int_0^1 dq\, 2q^{1-\eta} = \frac{-2}{2-\eta}\,. \tag{6.12}$$

Collecting the results one observes that terms containing odd powers of p cancel and up to $\mathcal{O}\left(p^4\right)$ one finds (4.22) with (4.23) and (4.24).

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